Solutions to Quiz 1, ECED 3300

Problem 1

a) By definition,

$$I(R) = \int d\mathbf{S} \cdot \mathbf{J},$$

On the surface of the sphere, $\mathbf{J} = \mathbf{a}_r J_0 R/R = J_0 \mathbf{a}_r$; $d\mathbf{S} = \mathbf{a}_r R^2 \sin\theta d\theta d\phi$, implying that $d\mathbf{S} \cdot \mathbf{J} = J_0 R^2 \sin\theta d\theta d\phi$. Thus,

$$I(R) = J_0 R^2 \underbrace{\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi}_{=4\pi} = 4\pi R^2 J_0.$$

By definition,

$$I = \frac{dQ}{dt} \Longrightarrow Q = \int_0^t dt I = 4\pi R^2 J_0 t.$$

b) The total current through the shell is given by

$$I = \underbrace{\int_1 d\mathbf{S} \cdot \mathbf{J}}_{\text{inner sphere}} + \underbrace{\int_2 d\mathbf{S} \cdot \mathbf{J}}_{\text{outer sphere}}.$$

On the inner sphere $\mathbf{J} = \mathbf{a}_r J_0 R_1/R$, $\mathbf{a}_n = -\mathbf{a}_r$ and on the outer sphere, $\mathbf{J} = \mathbf{a}_r J_0 R_2/R$, $\mathbf{a}_n = \mathbf{a}_r$. It follows that

$$I = 4\pi J_0 R_2^3/R - 4\pi J_0 R_1^3/R = 4\pi J_0 (R_2^3 - R_1^3)/R.$$

Problem 2

By definition,

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta \mathbf{a}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & \sin^2\theta/r \end{vmatrix} = \frac{1}{r^3}(2\cos\theta \mathbf{a}_r + \sin\theta \mathbf{a}_\theta)$$

Problem 3

As the cylinder is very long, translational symmetry along the axis, coupled with axial symmetry, allow us to use Ampère's law. The current density $\mathbf{J} = \mathbf{a}_z I/(\pi a^2)$, where we place the conductor such that its axis coincides with the z axis. By symmetry, we guess the field direction to be azimuthal, $\mathbf{H} = H\mathbf{a}_\phi$. Using circles as Ampèrian paths, we obtain

a)

$$H\,2\pi\rho = J\pi\rho^2 = \left(\frac{I}{\pi a^2}\right)\pi\rho^2 \Longrightarrow H = \frac{I\rho}{2\pi a^2}.$$

b)

$$H\,2\pi\rho = I \Longrightarrow H = \frac{I}{2\pi\rho}.$$

Thus,

$$\mathbf{H} = \begin{cases} \left(\frac{I\rho}{2\pi a^2}\right)\mathbf{a}_\phi, & \rho \leq a; \\ \left(\frac{I}{2\pi\rho}\right)\mathbf{a}_\phi, & \rho \geq a. \end{cases}$$